

FP2 S13 R

1. A transformation T from the z -plane to the w -plane is given by

$$w = \frac{z + 2i}{iz} \quad z \neq 0$$

The transformation maps points on the real axis in the z -plane onto a line in the w -plane.

Find an equation of this line.

(4)

$$wi z = z + 2i$$

$$wi z - z = 2i$$

$$z(wi - 1) = 2i$$

$$z = \frac{2i}{wi - 1} = \frac{-2}{-w - i} = \frac{2}{w + i} = \frac{2}{u + (v+1)i} \times \frac{(u - (v+1)i)}{(u - (v+1)i)}$$

$$z = \frac{2u - 2(v+1)i}{u^2 + (v+1)^2} \quad \therefore \text{real part maps to } \frac{2u}{u^2 + (v+1)^2}$$

$\Rightarrow$ Imaginary part is zero

$$\Rightarrow \frac{2(v+1)}{u^2 + (v+1)^2} = 0 \quad \Rightarrow \underline{v = -1}$$

alt $z = x + iy$ points on real axis $\Rightarrow$ Imaginary part is zero $\therefore z = x$

$$w = \frac{z + 2i}{iz} \Rightarrow w = \frac{x + 2i}{xi} = \frac{1}{i} + \frac{2}{x}$$

$$\Rightarrow w = \frac{2}{x} - i \Rightarrow u + iv = \left(\frac{2}{x}\right) + (-1)i$$

$$\Rightarrow u = \frac{2}{x} \quad \underline{\underline{v = -1}}$$

2. Use algebra to find the set of values of x for which

$$\frac{6x}{3-x} > \frac{1}{x+1}$$

(7)

$$\frac{6x(3-x)^2(x+1)^2}{\cancel{3-x}} > \frac{(x+1)^2(3-x)^2}{\cancel{x+1}}$$

$$6x(3-x)(x+1)^2 - (x+1)(3-x)^2 > 0$$

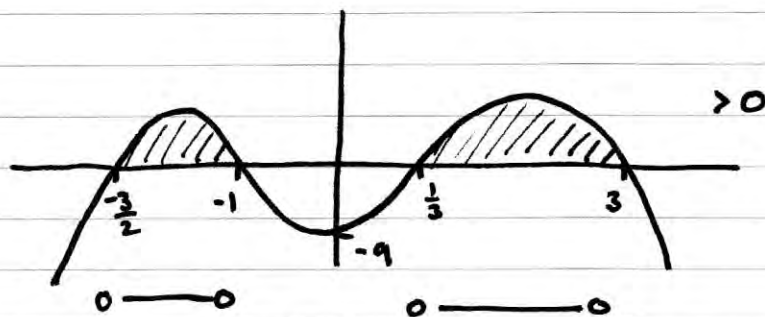
$$(3-x)(x+1)[6x(x+1) - (3-x)] > 0$$

$$(3-x)(x+1)(6x^2+6x-3+x) > 0$$

$$(3-x)(x+1)(6x^2+7x-3) > 0$$

$$(3-x)(x+1)(3x-1)(2x+3) > 0$$

$$3 \quad -1 \quad \frac{1}{3} \quad -\frac{3}{2}$$



$$\left(-\frac{3}{2} < x < -1\right) \cup \left(\frac{1}{3} < x < 3\right)$$

3. (a) Express $\frac{2}{(r+1)(r+3)}$ in partial fractions.

(2)

(b) Hence show that

$$\sum_{r=1}^n \frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}$$

(4)

(c) Evaluate $\sum_{r=10}^{100} \frac{2}{(r+1)(r+3)}$, giving your answer to 3 significant figures.

(2)

$$\begin{aligned} \text{a) } &= \frac{A}{r+1} + \frac{B}{r+3} \Rightarrow A(r+3) + B(r+1) = 2 \\ &\quad r = -1 \quad 2A = 2 \quad \underline{A = 1} \\ &\quad r = -3 \quad -2B = 2 \quad \underline{B = -1} \end{aligned}$$

$$= \frac{1}{r+1} - \frac{1}{r+3}$$

$$\begin{aligned} \text{b) } \sum_{r=1}^n \frac{2}{(r+1)(r+3)} &= \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \dots \\ &\quad + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \left(\frac{1}{n} - \frac{1}{n+2} \right) + \left(\frac{1}{n+1} - \frac{1}{n+3} \right) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} = \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3} \\ &= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)} = \frac{5n^2 + 25n + 30 - 6n - 18 - 6n - 12}{6(n+2)(n+3)} \end{aligned}$$

$$= \frac{5n^2 + 13n}{6(n+2)(n+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}$$

$$\text{c) } \sum_{r=10}^{100} \frac{2}{(r+1)(r+3)} = \frac{100(500+13)}{6(102)(103)} - \frac{9(45+13)}{6(11)(12)} \approx 0.155 \text{ (3sf)}$$

5. (a) Find, in the form $y = f(x)$, the general solution of the equation

$$\frac{dy}{dx} + 2y \tan x = \sin 2x, \quad 0 < x < \frac{\pi}{2}$$

(6)

Given that $y = 2$ at $x = \frac{\pi}{3}$

(b) find the value of y at $x = \frac{\pi}{6}$, giving your answer in the form $a + k \ln b$, where a and b are integers and k is rational.

(4)

$$\frac{dy}{dx} + (2 \tan x)y = \sin 2x$$

$$\text{IF} \Rightarrow f(x) = e^{\int \tan x} = (e^{\ln|\sec x|})^2 = \sec^2 x$$

$$\Rightarrow \sec^2 x \frac{dy}{dx} + \sec^2 x (2 \tan x)y = \sec^2 x \sin 2x$$

$$\frac{d}{dx}(y \sec^2 x) = \frac{2 \sin x \cos x}{\cos^4 x} = 2 \tan x$$

$$\Rightarrow y \sec^2 x = \int \tan x dx = 2 \ln|\sec x| + c$$

$$\therefore y = \frac{2 \ln|\sec x| + c}{\sec^2 x}$$

$$\text{b) } \left(\frac{\pi}{3}, 2\right) \quad 2 = \frac{2 \ln\left(\frac{1}{\cos \frac{\pi}{3}}\right) + c}{\left(\frac{1}{\cos \frac{\pi}{3}}\right)^2} \Rightarrow 2 = \frac{2 \ln 2 + c}{4} \\ \therefore c = 8 - 2 \ln 2.$$

$$\therefore y = \frac{2 \ln \left| \frac{\sec x}{2} \right| + 8}{\sec^2 x}$$

$$x = \frac{\pi}{6} \quad y = \frac{2 \ln\left(\frac{1}{2 \cos \frac{\pi}{6}}\right) + 8}{\left(\frac{1}{\cos \frac{\pi}{6}}\right)^2} = \frac{2 \ln\left(\frac{1}{\sqrt{3}}\right) + 8}{\left(\frac{2}{\sqrt{3}}\right)} = \frac{2 \ln(3)^{-\frac{1}{2}} + 8}{\left(\frac{2}{\sqrt{3}}\right)}$$

$$y = 6 - \frac{3}{4} \ln 3$$

6. The complex number $z = e^{i\theta}$, where θ is real.

(a) Use de Moivre's theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

where n is a positive integer.

(2)

(b) Show that

$$\cos^5 \theta = \frac{1}{16} (\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$$

(5)

(c) Hence find all the solutions of

$$\cos 5\theta + 5\cos 3\theta + 12\cos \theta = 0$$

in the interval $0 \leq \theta < 2\pi$

(4)

$$\begin{aligned} \text{a) } z &= \cos \theta + i \sin \theta & z^n &= (\cos \theta + i \sin \theta)^n \\ & & z^n &= \cos n\theta + i \sin n\theta \end{aligned}$$

$$\begin{aligned} z^n + \frac{1}{z^n} &= z^n + z^{-n} = (\cos n\theta + i \sin n\theta) + (\cos(-n\theta) + i \sin(-n\theta)) \\ &= (\cos n\theta + i \sin n\theta) + (\cos n\theta - i \sin n\theta) \\ &= 2\cos(n\theta) \end{aligned}$$

$$\text{b) } \left(z + \frac{1}{z}\right)^5 = (2\cos \theta)^5 = 32\cos^5 \theta$$

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & 1 & 2 & 1 & & \\ & & 1 & 3 & 3 & 1 & \\ & & 1 & 4 & 6 & 4 & 1 \\ & & 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

$$\begin{aligned} \left(z + \frac{1}{z}\right)^5 &= z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) + 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) + \left(\frac{1}{z^5}\right) \\ &= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right) \end{aligned}$$

$$\Rightarrow 32\cos^5 \theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos \theta$$

$$\therefore \cos \theta = \frac{1}{16} (\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$$

$$\text{c) } 16\cos^5 \theta = \cos 5\theta + 5\cos 3\theta + 10\cos \theta \quad \therefore 16\cos^5 \theta + 2\cos \theta = \cos 5\theta + 5\cos 3\theta + 12\cos \theta$$

$$\Rightarrow 2\cos \theta (8\cos^4 \theta + 1) = 0 \quad \therefore \cos \theta = 0 \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

no solution

7. (a) Find the value of λ for which $\lambda t^2 e^{3t}$ is a particular integral of the differential equation

$$\frac{d^2 y}{dt^2} - 6 \frac{dy}{dt} + 9y = 6e^{3t}, \quad t \geq 0 \quad (5)$$

- (b) Hence find the general solution of this differential equation.

(3)

Given that when $t = 0$, $y = 5$ and $\frac{dy}{dt} = 4$

- (c) find the particular solution of this differential equation, giving your solution in the form $y = f(t)$.

(5)

a) $y = \lambda t^2 e^{3t}$
 $y' = 3\lambda t^2 e^{3t} + 2\lambda t e^{3t} = (3\lambda t^2 + 2\lambda t) e^{3t}$
 $y'' = 3(3\lambda t^2 + 2\lambda t) e^{3t} + (6\lambda t + 2\lambda) e^{3t} = (9\lambda t^2 + 12\lambda t + 2\lambda) e^{3t}$

$$y'' - 6y' + 9y = \begin{pmatrix} 9\lambda t^2 + 12\lambda t + 2\lambda \\ -18\lambda t^2 - 12\lambda t \\ +9\lambda t^2 \end{pmatrix} e^{3t} = 6e^{3t} \quad \therefore \begin{matrix} 2\lambda = 6 \\ \lambda = 3 \end{matrix}$$

$$y_{PI} = 3t^2 e^{3t}$$

b) $y = Ae^{mt}$
 $y' = Am e^{mt}$
 $y'' = Am^2 e^{mt}$
 $y'' - 6y' + 9y = 0$
 $Ae^{mt} (m^2 - 6m + 9) = 0$
 $\neq 0 \quad = 0 \quad (m-3)^2 = 0 \quad \therefore m = 3$

$$y_{CF} = (A + Bt) e^{3t} \quad \therefore y_{GS} = (A + Bt) e^{3t} + 3t^2 e^{3t} = (A + Bt + 3t^2) e^{3t}$$

$$t = 0, y = 5 \Rightarrow A = 5$$

$$y = (5 + Bt + 3t^2) e^{3t}$$

$$t = 0, y' = 4$$

$$y' = 3(5 + Bt + 3t^2) e^{3t} + (B + 6t) e^{3t}$$

$$4 = 15 + B \quad \therefore B = -11$$

$$\therefore y = (5 - 11t + 3t^2) e^{3t}$$

8.

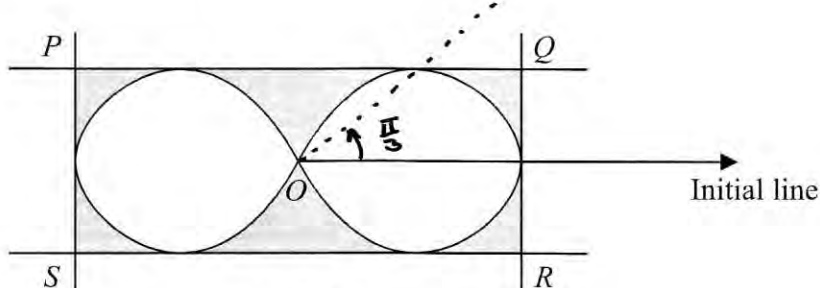


Figure 1

Figure 1 shows a closed curve C with equation

$$r = 3(\cos 2\theta)^{\frac{1}{2}}, \quad \text{where } -\frac{\pi}{4} < \theta \leq \frac{\pi}{4}, \quad \frac{3\pi}{4} < \theta \leq \frac{5\pi}{4}$$

The lines PQ , SR , PS and QR are tangents to C , where PQ and SR are parallel to the initial line and PS and QR are perpendicular to the initial line. The point O is the pole.

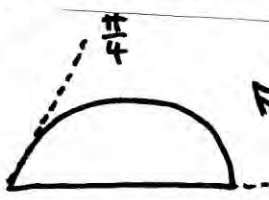
(a) Find the total area enclosed by the curve C , shown unshaded inside the rectangle in Figure 1.

(4)

(b) Find the total area of the region bounded by the curve C and the four tangents, shown shaded in Figure 1.

(9)

a)



$$\begin{aligned} \text{unshaded} &= 4 \times \frac{1}{2} \int_0^{\pi/4} (3 \cos 2\theta)^{\frac{1}{2}} \cdot 3 \cos 2\theta \, d\theta \\ &= 4 \times \frac{9}{2} \int_0^{\pi/4} \cos 2\theta \, d\theta = 18 \left[\frac{1}{2} \sin 2\theta \right]_0^{\pi/4} = 18 \left(\frac{1}{2} - 0 \right) = 9 \end{aligned}$$

b) at $\theta=0$ $r=3 \therefore \overline{SR}=6$.

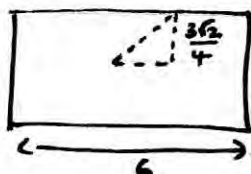
Along PQ tangent to curve is parallel to initial line, $\frac{dy}{d\theta}=0$

$$y = r \sin \theta = 3(\cos 2\theta)^{\frac{1}{2}} \times \sin \theta$$

$$\frac{dy}{d\theta} = 3(\cos 2\theta)^{\frac{1}{2}} \times (\cos \theta - \frac{3}{2}(\cos 2\theta)^{-\frac{1}{2}} \times 2 \sin 2\theta \times \sin \theta)$$

$$\begin{aligned} \frac{dy}{d\theta} = 0 &\Rightarrow 3(\cos 2\theta) \times \cos \theta = 3(2 \sin \theta \cos \theta) \sin \theta \\ &\Rightarrow (1 - 2 \sin^2 \theta) = 2 \sin^2 \theta \Rightarrow \sin^2 \theta = \frac{1}{4} \Rightarrow \sin \theta = \frac{1}{2} \end{aligned}$$

$$y = 3(\cos 2\theta)^{\frac{1}{2}} \times \sin \theta \quad \theta = \frac{\pi}{6} \quad y = 3 \sqrt{\frac{1}{2}} \times \frac{1}{2} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$$



$$\text{Rectangle} = 6 \times \frac{3\sqrt{2}}{2} = 9\sqrt{2}$$

$$\therefore \text{unshaded} = 9\sqrt{2} - 9$$